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2000 J. Phys. A: Math. Gen. 33 731

(http://iopscience.iop.org/0305-4470/33/4/307)

J. Phys. A: Math. Gen. **33** (2000) 731–740. Printed in the UK PII: S0305-4470(00)06245-4

Bose–Einstein condensation in an exactly solvable model of strongly interacting bosons

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Received 22 July 1999

Abstract. An exact algebraic solution is presented for the spectrum of the Bose–Hubbard model for pure systems in the limit of infinite-range hopping and infinite on-site repulsion. This strongly interacting boson system is shown to exhibit Bose–Einstein condensation into the lowest unperturbed single-particle state, with a transition temperature which, for any density of bosons, is always lower than that in the absence of interactions.

1. Introduction

Bose–Einstein condensation has once again become a subject of considerable current interest. It is most familiar in the case of non-interacting bosons, where below the transition temperature a macroscopic number of bosons occupy the same single-particle state. It is conceptually straightforward to also apply the notion of condensation to weakly interacting bosons. However, in strongly interacting boson systems, where the very idea of a single-particle state tends to lose its significance, it is far less evident how to interpret physically the phenomenon of condensation. The hard-sphere Bose gas in three dimensions is an example of a strongly interacting model which has been carefully studied by Monte Carlo methods [1]. It was found that the Bose–Einstein condensation persists in the presence of the hard-sphere repulsion, the interaction producing an increase in the transition temperature at low densities and a decrease at high densities.

In many fields of physics, exactly solvable models have proved useful in elucidating the physics of a system and in evaluating the reliability of approximate methods of solution. Although the assumptions required to make the models solvable generally also make them rather unrealistic or unphysical, the lessons learned from their study can frequently be taken over to more realistic contexts. In this spirit, an exactly solvable model of strongly interacting bosons could be helpful in understanding the nature of Bose–Einstein condensation in strongly interacting boson systems.

A fair degree of attention has lately been lavished on the Bose–Hubbard model, which is a simple and convenient representation of a system of interacting bosons [2]. The emphasis has, naturally, been mainly on the thermodynamic properties of the model and considerable progress has been made in understanding its phase diagram. It is well known that the model becomes equivalent to an XXZ Heisenberg model when the on-site repulsion becomes infinite, thus ensuring that each lattice site is occupied by at most one boson [3]. An exact thermodynamic solution has been presented in the limit of infinite-range hopping and finite on-site repulsion [4]. When both limits are taken, allowing for infinite-range hopping with infinite on-site repulsion,

0305-4470/00/040731+10$30.00

the Bose–Hubbard Hamiltonian turns out to be exactly solvable by algebraic methods. The complete spectrum, with its associated degeneracies, and even the exact eigenvectors, can be obtained analytically, as will be shown in section 2. The exact solution can then be used to study the phenomenon of Bose–Einstein condensation in this not very physical model.

In its most general form, the Bose–Hubbard model is defined by the second quantized Hamiltonian

$$H_g = \sum_i (\epsilon_i - \mu)\hat{n}_i - t \sum_{\langle i,j \rangle} a_i^\dagger a_j + U \sum_i \hat{n}_i(\hat{n}_i - 1) \tag{1}$$

where i and j are site labels, $a_i^\dagger$ and a_i are boson creation and destruction operators, $\hat{n}_i = a_i^\dagger a_i$ is the number operator at site i and the notation $\langle i, j \rangle$ denotes nearest-neighbour pairs of sites. Here ϵ_i is an on-site energy which allows for the presence of disorder, μ is the chemical potential associated with changes in the overall number of bosons, t is the hopping strength between neighbouring sites (usually taken as positive) and U is the on-site interaction between bosons (positive for repulsive interactions, negative for attractive interactions). It should be noted that the Hamiltonian H_g strictly conserves the total number of bosons. The case of interest here is obtained by specializing H_g by the imposition of the following conditions: (a) no disorder (the so-called pure case), $\epsilon_i = 0$; (b) unlimited hopping, $\langle i, j \rangle \to i \neq j$; (c) infinite on-site repulsion, $U \to \infty$. Since the chemical potential term is proportional to the (conserved) total number of bosons, it amounts solely to a shift in the zero of energy, when a fixed number of bosons is being considered. It will thus be omitted, (d) $\mu = 0$. The Hamiltonian to be studied thus becomes

$$H = -t \sum_{i \neq j} a_i^\dagger a_j \tag{2}$$

with the added constraint that each site is at most singly occupied, as a consequence of the infinite on-site repulsion.

In the specialized form of equation (2), with the sum covering all distinct pairs of sites, this limit of the Bose–Hubbard model is independent of the dimension of the system and of the lattice type. In fact, no lattice structure at all is required—it is sufficient for the sites to be denumerable. It is assumed that there are n bosons distributed among N sites, with the number density defined by the filling fraction $f = n/N$. Since no site can be multiply occupied, $n \leqslant N$ and $0 \leqslant f \leqslant 1$. The solution of the model will be derived for fixed, finite values of the boson number n and the number of sites N. For thermodynamic applications, the relevant limit is $n \to \infty$, $N \to \infty$ with f constant.

2. Algebraic solution

The model defined by the Hamiltonian (2) can be solved exactly by a method very similar to that used recently to solve the fermion version of the same limit of the regular Hubbard model [5]. At each site i, a pair of states $|i0\rangle$ and $|ib\rangle$ is defined, where the former represents a vacant site and the latter a site occupied by a single boson. They have the self-evident property

$$\langle ix|jy\rangle = \delta_{ij}\delta_{xy} \qquad (x, y = 0, b; i, j = 1, 2, \ldots, N). \tag{3}$$

These states can be used to construct at each site the four Hubbard projectors

$$X_{00}^{(i)} = |i0\rangle\langle i0| \tag{4a}$$

$$X_{0b}^{(i)} = |i0\rangle\langle ib| \tag{4b}$$

$$X_{b0}^{(i)} = |ib\rangle\langle i0| \tag{4c}$$

$$X_{bb}^{(i)} = |ib\rangle\langle ib|. \tag{4d}$$

The prohibition of multiple occupancy of the site is implemented by the constraint

$$X_{00}^{(i)} + X_{bb}^{(i)} = 1. \tag{5}$$

There are thus only three independent Hubbard operators at each site. The orthogonality property, equation (3), ensures that they satisfy the commutators

$$[X_{bb}^{(i)}, X_{b0}^{(j)}] = \delta_{ij} X_{b0}^{(i)} \tag{6a}$$

$$[X_{bb}^{(i)}, X_{0b}^{(j)}] = -\delta_{ij} X_{0b}^{(i)} \tag{6b}$$

$$[X_{b0}^{(i)}, X_{0b}^{(j)}] = \delta_{ij}(X_{bb}^{(i)} - X_{00}^{(i)}) = \delta_{ij} 2\big(X_{bb}^{(i)} - \tfrac{1}{2}\big) \tag{6c}$$

where the constraint of equation (5) has been used in equation (6*c*). As a result, the set of three global operators

$$L_+ = \sum_i X_{b0}^{(i)} \tag{7a}$$

$$L_- = \sum_i X_{0b}^{(i)} \tag{7b}$$

$$L_0 = \sum_i \big(X_{bb}^{(i)} - \tfrac{1}{2}\big) = \hat{n} - N/2 \tag{7c}$$

satisfies the algebra of $SU(2)$,

$$[L_0, L_\pm] = \pm L_\pm \tag{8a}$$

$$[L_+, L_-] = 2L_0. \tag{8b}$$

Note the identification $X_{bb}^{(i)} = \hat{n}_i$ and the definition of the total number operator $\hat{n} = \sum_i \hat{n}_i$.

All that remains is to note that the Bose–Hubbard Hamiltonian (2) can be transcribed as

$$H = -t \sum_{i,j} a_i^\dagger a_j + t\hat{n} = -t \sum_{i,j} X_{b0}^{(i)} X_{0b}^{(j)} + t\hat{n} = -t(L_+ L_- - \hat{n}). \tag{9}$$

Using the properties of $SU(2)$, this can be further rewritten in the highly suggestive form

$$H = -t\big[\vec{L}^2 - N/2(N/2+1) + \hat{n}(N-\hat{n})\big] \tag{10}$$

where $\vec{L}^2$ is the usual Casimir operator of $SU(2)$. The eigenstates of this Hamiltonian are clearly the $SU(2)$ states $|\ell, \ell_z\rangle$, with corresponding eigenvalues

$$E = -t[\ell(\ell+1) - \ell_z^2 - N/2]. \tag{11}$$

For given N, this spectrum is symmetric in n about the midpoint $n = N/2$.

It is easily checked that the single-site states $|i0\rangle$ and $|ib\rangle$ form an $\ell = \frac{1}{2}$ doublet of this $SU(2)$—the former having $\ell_z = -\frac{1}{2}$ and the latter $\ell_z = +\frac{1}{2}$—so that the N-site states must be of the form $(\frac{1}{2})^N$. The accessible values of ℓ therefore lie between zero

and $N/2$. For a fixed value of n, the value of ℓ_z is fixed at $n - N/2$, so ℓ cannot be less than $|n - N/2|$. By inspection, the lowest eigenstates will have $\ell = N/2$, higher eigenstates having ℓ values decreasing in unit steps until the highest eigenstates are attained, with $\ell = |n - N/2|$.

The degeneracy of the eigenstates is also easily determined by standard $SU(2)$ techniques. The value of ℓ_z is determined by the number of bosons on the lattice, $\ell_z = n - N/2$, so the number of states of a given ℓ_z is simply the number of ways of distributing n particles over N sites, allowing at most one particle per site. There are thus $\binom{N}{n} = \binom{N}{\ell_z+N/2}$ states of a given ℓ_z. It follows that there are $\binom{N}{\ell+N/2} - \binom{N}{\ell+1+N/2}$ states of given ℓ, each associated with $2\ell + 1$ substates having $\ell_z = -\ell, \ldots, \ell$. The degeneracy of the eigenstate $|\ell\ell_z\rangle$ of the Hamiltonian (2) for a given number $n = \ell_z + N/2$ of bosons on N sites is therefore

$$g_N(\ell) = \frac{N!(2\ell + 1)}{(N/2 - \ell)!(N/2 + \ell + 1)!}. \tag{12}$$

To summarize, the exact spectrum of the Hamiltonian (2) for n bosons on N sites is given by equation (11), with degeneracy given by equation (12), where $\ell_z = n - N/2$ and ℓ takes all values from $N/2$ down to $|\ell_z|$, in unit steps.

The same solution may be obtained by a very elegant alternate method, originally developed to deal with the fermion version of the same model [6]. It is based on the observation that the Hamiltonian (2) is invariant under all permutations of the site labels. Its eigenstates can therefore be classified according to the appropriate irreducible representations of the symmetric group on N objects, S_N. These representations are specified by Young diagrams. Since the present system involves two distinct bosons—occupied sites and vacancies—the relevant diagrams have at most two rows, with a total of N boxes. The first n boxes of the upper row are labelled as occupied sites, the remaining boxes of the upper row and all boxes of the lower row as vacancies. The lower row cannot be longer than n boxes, since two vacancies cannot occupy the same column. By denoting the number of boxes in the upper row as $N/2 + \ell$ and the number in the lower row as $N/2 - \ell$, each relevant representation can be labelled by the single quantum number ℓ, which is easily seen to have the same range of allowed values as in the $SU(2)$ approach. The number of distinct states in the irreducible representation labelled by ℓ is found, from standard formulae for the symmetric group, to be precisely the degeneracy presented in equation (12). The eigenvalues can be found by the algebraic methods described in [6]. In addition, this method allows the explicit construction of the eigenstates by application of Young projectors to simple product states, as outlined in [6].

It is manifestly clear that this combinatoric treatment of the model is particle–hole symmetric—occupied and vacant states may be exchanged throughout the above discussion. This is true also of the $SU(2)$ treatment, but there it is less transparent. It is thus sufficient to restrict detailed discussions of the model to the case $n \leqslant N/2$, using particle–hole symmetry to complete the picture.

3. Comparison of perturbed and unperturbed cases

The Hamiltonian of equation (2), with no constraint on the number of bosons at a site, may be considered as an unperturbed version of the model solved in the previous section. The strong repulsive interaction between bosons at the same site enforces the prohibition on multiple occupancy of a site and produces the perturbed form. It is instructive to compare the solutions of the perturbed and unperturbed problems.

The unperturbed system is a pure one-body problem and is easily solved by Fourier transformation,

$$a_\alpha^\dagger = (1/\sqrt{N}) \sum_{k=1}^{N} \mathrm{e}^{2\pi \mathrm{i}\alpha k/N} a_k^\dagger \tag{13}$$

where $k = 1, 2, \ldots, N$ labels sites and $\alpha = 0, 1, 2, \ldots, N-1$ labels unperturbed single-particle states. The lowest such state, created by the operator $a_0^\dagger = \sum_{i=1}^{N} a_i^\dagger$, has an energy $-(N-1)t$. All the rest of these states are degenerate at the energy t. The unperturbed Hamiltonian then takes the form

$$H_0 = -t(N a_0^\dagger a_0 - \hat{n}) = -t(N\hat{n}_0 - \hat{n}) \tag{14}$$

with eigenvalues determined solely by the occupation numbers n_0 and n. The eigenstates are defined by the occupation numbers of all unperturbed single-particle states and their degeneracies are the result of the degeneracy of all but the lowest single-particle state, which is shifted down by Nt from all the others. The unperturbed spectrum, for given N and n, is a set of equally spaced states, labelled by the quantum number n_0, of energy

$$E_{n_0}^{(0)} = -t(Nn_0 - n) \qquad n_0 = n, n-1, n-2, \ldots, 1, 0 \tag{15}$$

and degeneracy

$$g_N^{(0)}(n_0) = \frac{(N+n-n_0-2)!}{(n-n_0)!(N-2)!} \tag{16}$$

the number of distinct ways of distributing $n - n_0$ bosons among $N-1$ single-particle states.

For later applications, it is convenient to relabel the eigenstates and eigenenergies in terms of $\nu_0 = n - n_0$, the number of bosons in excited single-particle states. The energies are then

$$E_{\nu_0}^{(0)} = -t(N-1)n + Nt\nu_0 = E_0^{(0)} + Nt\nu_0 \qquad \nu_0 = 0, 1, 2, \ldots, n \tag{17}$$

with degeneracies

$$g_N^{(0)}(\nu_0) = \frac{(N+\nu_0-2)!}{\nu_0!(N-2)!}. \tag{18}$$

The ground-state wavefunction is

$$|\Psi_0^{(0)}\rangle = (1/\sqrt{n!})(a_0^\dagger)^n |0\rangle \tag{19}$$

where $|0\rangle$ is the boson vacuum, i.e. the empty lattice. This state is non-degenerate, $g_N^{(0)}(0) = 1$.

The perturbed spectrum, obtained when the infinite on-site repulsion is implemented by way of the prohibition on multiple occupancy of any lattice site, is given in the previous section in terms of the $SU(2)$ quantum number ℓ. For comparison with the unperturbed spectrum, and also for further applications, it is convenient to relabel the perturbed results in terms of the quantum number $\nu = N/2 - \ell$, which defines the deviation of ℓ from its maximum (ground-state) value. The energies (11) are then

$$E_\nu = -tn(N-n) + t(N+1-\nu)\nu = E_0 + t(N+1-\nu)\nu \qquad \nu = 0, 1, 2, \ldots, n \tag{20}$$

with degeneracies

$$g_N(\nu) = \frac{N!(N+1-2\nu)}{\nu!(N+1-\nu)!}. \tag{21}$$

Particle–hole symmetry has been exploited to limit n to values less than or equal to $N/2$, i.e. the fractional occupancy of the lattice is limited to $f \leqslant \frac{1}{2}$. (The results for values $\frac{1}{2} \leqslant f \leqslant 1$ are immediately obtained from the symmetry about half filling.)

The wavefunction of the perturbed ground state can easily be checked to be

$$|\Psi\rangle = \sqrt{\frac{(N-n)!}{n!N!}} \sum_{\{i_1 i_2 \ldots i_n\}} a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle \tag{22}$$

where the sum is over all sets of n distinct site labels, i.e. no site is occupied more than once in any term. This state, too, is non-degenerate, $g_N(0) = 1$.

The states of the unperturbed spectrum are equally spaced, with a fixed energy difference of Nt between any two successive states. The spacing of the perturbed spectrum, however, gradually decreases with distance from the ground state, the gap between the νth excited state and its successor being only $(N-2\nu)t$. (The first excited state in the perturbed spectrum is at the same excitation energy as the first excited unperturbed state.) The degeneracy of corresponding states—the rth excited state in each spectrum, say—is greater in the unperturbed than in the perturbed spectrum. In fact, the ratio of the degeneracy of a typical excited perturbed state to that of the corresponding unperturbed state becomes exponentially small for large N. This is to be expected, since the effect of the interaction is to exclude otherwise permitted configurations, and is also reflected in the total number of allowed states, which is $(N+n-1)!/n!(N-1)!$ in the unperturbed case and $N!/n!(N-n)!$ in the perturbed case. For large N and n, with fixed $f = n/N$, the ratio of the total number of perturbed to the total number of unperturbed states is $\sqrt{(1+f)/(1-f)}\left[(1-f)^{1-f}(1+f)^{1+f}\right]^{-N}$, where the quantity in the square brackets is greater than unity for all $f > 0$.

The overlap between the ground-state wavefunctions (19) and (22) can be computed by iteration in the number of bosons n and is found to be

$$\langle\Psi_0|\Psi\rangle = \sqrt{\frac{N!}{(N-n)!N^n}}. \tag{23}$$

For large N and n, with fixed $f = n/N$, Stirling's approximation to the factorials gives

$$\langle\Psi_0|\Psi\rangle \sim \left[\frac{\mathrm{e}^{-f}}{(1-f)^{1-f}}\right]^{N/2} \tag{24}$$

which is generally very small, except when $f \simeq 0$. Numerical evaluation of equation (23) shows that the square of the overlap between the perturbed and unperturbed ground-state wavefunctions falls rapidly from its value of unity at $f = 0$ to zero, the width at half maximum (i.e. the value of f at which the squared overlap is 0.5) being well represented as $1.18/\sqrt{N}$.

Finally, the average number of bosons in the lowest unperturbed single-particle state is given by the expectation value of the number operator $\hat{n}_0$. Because of the special form of the Hamiltonian (14), this is very simply related to the expectation value of the Hamiltonian. In any energy eigenstate it is quickly written down. In the unperturbed ground state it takes the value $\langle\hat{n}_0\rangle = n$, as it must, while in the perturbed ground state it takes the value $\langle\hat{n}_0\rangle = n - n(n-1)/N$. In the limit of large N and n with fixed f, the average fraction of bosons in the lowest single-particle state is unity in the unperturbed case and $1-f$ in the perturbed case, in the ground state of the system. Although the infinite on-site repulsion introduces strong correlations between the particles, the lowest unperturbed single-particle state remains macroscopically occupied in the ground state of the perturbed system, though the occupancy is reduced by the filling fraction f.

4. Thermodynamics

The thermodynamic limit of the model is defined by $N \to \infty$, $n \to \infty$, $f = n/N$ fixed. As will be clear from a glance at equations (17) and (20), in order for this limit to be sensible (extensive energies, for instance) it is necessary to rescale the hopping strength t. The rescaled hopping strength is defined by

$$\kappa = Nt. \tag{25}$$

It is also convenient to shift the zero of energy so that the system ground state lies at zero energy. This simply requires setting $E_0^{(0)} \to 0$ in equation (17) and $E_0 \to 0$ in equation (20). Finally, since κ is now the only parameter of the model with dimensions of energy, it sets the energy scale of all results and can conventionally be taken as the unit of energy, $\kappa = 1$. Similarly, temperature will be measured on an energy scale by setting Boltzmann's constant $k_B = 1$, so the dimensionless temperature T should be interpreted as $k_B T/\kappa$.

The perturbed form of the model cannot be dealt with in terms of single-particle degrees of freedom, because of the strong correlations induced by the prohibition of multiple occupancy of any site. It is therefore not useful to work in terms of the grand canonical ensemble, so that a canonical ensemble treatment is preferred.

The partition function for the perturbed system is

$$Z = \sum_{\nu=0}^{n} g_N(\nu)\, \mathrm{e}^{-E_\nu/T} = \sum_{\nu=0}^{n} \frac{N!(N+1-2\nu)}{\nu!(N+1-\nu)!}\, \mathrm{e}^{-\nu[1-(\nu-1)/N]/T}. \tag{26}$$

By applying Stirling's approximation to the factorials, a typical term in the sum can be written, for large n and N, as $\left[\mathrm{e}^{-\phi(1-\phi)/T}/\phi^\phi(1-\phi)^{1-\phi}\right]^N (1-2\phi)/\left[(1-\phi)\sqrt{2\pi N\phi(1-\phi)}\right]$, where $\phi = \nu/N$ and $0 \leqslant \phi \leqslant f < \frac{1}{2}$. (Recall that fractional occupancies greater than $f = \frac{1}{2}$ are treated by particle–hole symmetry.) The ratio of the $(\nu+1)$th term to its predecessor is similarly approximated by $\rho = \mathrm{e}^{-(1-2\phi)/T}(1-\phi)/\phi$. This ratio is generally larger than unity for small ϕ, falling as ϕ increases, so the terms of the sum typically increase steadily, until $\rho = 1$. The partition function is thus dominated by the largest term, in which ϕ takes the value $\phi^{(m)}$ for which $\rho = 1$. For a given temperature T, this dominant $\phi^{(m)}$ is determined implicitly as the solution of the equation

$$\mathrm{e}^{-(1-2\phi)/T} = \phi/(1-\phi). \tag{27}$$

Since the sum over ν is limited by n, the quantity ϕ is limited by f and there is a critical temperature at which $\phi^{(m)} = f$. Above this temperature, the partition function is always dominated by the last term in the sum. It may be concluded that the perturbed system has a critical transition temperature, given by

$$T_c = \frac{1-2f}{\ln(1-f) - \ln f}. \tag{28}$$

The free energy is given by $F = -T \ln Z$. Upon approximating Z by its dominant term (again recalling that N is very large), it is found that $F = N[(\phi^{(m)})^2 + T\ln(1-\phi^{(m)})]$ below T_c, with $\phi^{(m)}$ given by the solution of equation (27), while $F = N\{f(1-f) + T[f\ln f + (1-f)\ln(1-f)]\}$ above T_c. With the usual identification $U = -T^2(\partial/\partial T)(F/T)$, it is easily shown that the internal energy is $U = N\phi^{(m)}(1-\phi^{(m)})$ below T_c and $U = Nf(1-f)$ above T_c. This is continuous, but with a discontinuous derivative, at the critical temperature. The entropy of the system can also be computed, from the standard expression $S = -(\partial F/\partial T)$. It is found to be $S = -N[\phi^{(m)}\ln\phi^{(m)} + (1-\phi^{(m)})\ln(1-\phi^{(m)})]$ below T_c and $S = -N[f\ln f + (1-f)\ln(1-f)]$ above T_c.

A similar treatment may be given for the unperturbed system. The canonical partition function for the unperturbed system is given by

$$Z^{(0)} = \sum_{\nu_0=0}^{n} g_N^{(0)}(\nu_0)\, \mathrm{e}^{-E_{\nu_0}^{(0)}/T} = \sum_{\nu_0=0}^{n} \frac{(N+\nu_0-2)!}{\nu_0!(N-2)!}\, \mathrm{e}^{-\nu_0/T}. \tag{29}$$

Again applying Stirling's approximation to the factorials, a typical term in the sum can be written, for large n and N, as $\sqrt{(1+\phi_0)/2\pi N\phi_0}\,[\mathrm{e}^{-\phi_0/T}(1+\phi_0)^{1+\phi_0}/\phi_0^{\phi_0}]^N$, where $\phi_0 = \nu_0/N$. The ratio of the (ν_0+1)th term to its predecessor is similarly approximated by $\rho_0 = \mathrm{e}^{-\phi_0/T}(1+\phi_0)/\phi_0$. This ratio is generally larger than unity for small ϕ_0, falling as ϕ_0 increases, so the terms of the sum typically increase steadily, until $\rho_0 = 1$. The partition function is thus dominated by the largest term, for which $\rho_0^{(m)} = 1$ or $\phi_0^{(m)} = 1/(\mathrm{e}^{1/T}-1)$. However, ϕ_0 can never exceed f, since the upper limit on the sum over ν_0 is n, so there exists a critical temperature at which $1/(\mathrm{e}^{1/T_c^{(0)}}-1) = f$, above which the partition function is dominated by the last term in the sum. The transition occurs at the critical temperature $T_c^{(0)}$ for which

$$\mathrm{e}^{-1/T_c^{(0)}} = f/(1+f) \quad \Longrightarrow \quad T_c^{(0)} = 1/[\ln(1+f) - \ln f]. \tag{30}$$

The free energy of the unperturbed system is given by $F^{(0)} = -T\ln Z^{(0)}$. Upon approximating $Z^{(0)}$ by its dominant term (again recalling that N is very large), it is found that $F^{(0)} = NT\ln(1-\mathrm{e}^{-1/T})$ below $T_c^{(0)}$, while $F^{(0)} = n - NT[(1+f)\ln(1+f) - f\ln f]$ above $T_c^{(0)}$. The internal energy is $U^{(0)} = N/(\mathrm{e}^{1/T}-1)$ below $T_c^{(0)}$ and $U^{(0)} = n$ above $T_c^{(0)}$. The entropy of the system is found to be $S^{(0)} = N/[T(\mathrm{e}^{1/T}-1)] - N\ln(1-\mathrm{e}^{-1/T})$ below $T_c^{(0)}$ and $S^{(0)} = N[(1+f)\ln(1+f) - f\ln f]$ above $T_c^{(0)}$.

It may be concluded that the perturbed system shows the same kind of transition as does the unperturbed system, though at a different critical temperature. Both the perturbed and the unperturbed system may also be treated in the grand canonical ensemble. Although the treatment differs in detail from the above canonical treatment, the results are exactly the same in both approaches.

Both the perturbed and unperturbed systems show a transition at critical temperatures given by equations (28) and (30). The value of the critical temperature depends on the fractional occupancy $f = n/N$ of the lattice sites by the bosons, but for all values of f in the range $0 < f < \frac{1}{2}$ common to both systems, the critical temperature is lower in the perturbed than in the unperturbed system. The strong repulsion between bosons at the same site requires that the temperature be lowered in order to precipitate the transition. The ratio $T_c/T_c^{(0)} < 1$ for all f in the common range and decreases with increasing f, though it tends to unity as f approaches zero.

The nature of the transition is clarified by considering the average occupancy of the unperturbed single-particle ground state at various temperatures, $\overline{n_0}(T)$. Since the Hamiltonian (14) is expressed directly in terms of the number operator $\hat{n}_0$, this quantity can be easily transcribed from the average internal energy. Because of the rescaling of t shown in equation (25) and the shifts in zero of energy described immediately below that equation, the results for $U^{(0)}$ and U in the two previous subsections imply that, in the unperturbed system, $\overline{n_0^{(0)}}(T)/n = 1 - 1/f(\mathrm{e}^{1/T}-1)$ for $T < T_c^{(0)}$ and $\overline{n_0^{(0)}}(T)/n = 0$ for $T > T_c^{(0)}$; while in the perturbed system, $\overline{n_0}(T)/n = 1 - f - \phi^{(m)}(1-\phi^{(m)})/f$ for $T < T_c$ and $\overline{n_0}(T)/n = 0$ for $T > T_c$. For both the perturbed and the unperturbed systems, the relative occupancy of the unperturbed single-particle ground state is essentially constant for low temperatures, at the system ground state value (unity in the unperturbed system and $1-f$ in the perturbed system, see the end of section 3). As the temperature increases, this relative occupancy begins

to fall steadily, going to zero at and beyond the critical temperature. It is thus a reasonable candidate for an order parameter in this transition, which can be interpreted as a Bose–Einstein condensation. This is true even though the perturbed system exhibits strong correlations among the particles, which are forbidden to multiply occupy any site. The order parameter goes to zero at a lower temperature and with a steeper slope in the perturbed system than in the unperturbed system. (The slope at which zero is approached at the critical temperature is $-(1+f)\ln^2[f/(1+f)]$ in the unperturbed system and

$$\frac{-(1-f)\ln^2[f/(1-f)]}{1+2f(1-f)\ln[f/(1-f)]/(1-2f)}$$

in the perturbed system.) Since this slope is finite in both cases, both systems have a critical exponent of unity.

It is interesting to note that a hard-sphere Bose gas in three dimensions shows rather different behaviour from that obtained here. The current model is essentially a hard-sphere Bose lattice gas, but the unlimited hopping, which makes the results independent of dimensions and of lattice type, has a significant effect on the critical temperature for Bose–Einstein condensation. For the three-dimensional hard-sphere Bose gas it was found [1] that the repulsive interactions increase the critical temperature at low densities, but decrease it at high densities. In the present model, the infinite on-site repulsion reduces the critical temperature at all fractional occupancies of the lattice.

5. Summary and conclusions

It has been shown that the Bose–Hubbard model, when specialized to the limit of infinite-range hopping and infinite on-site repulsion, can be exactly and analytically solved in terms of an $SU(2)$ algebraic structure. The solution is independent of the dimension of the system and of the lattice type, and indeed holds even in the absence of any ordered lattice, as long as the sites are denumerable. For a fixed number n of bosons on a fixed number N of sites, the complete spectrum of the model has been given, together with the degeneracy of the levels, and it has been indicated how the eigenstates may be determined.

In the absence of on-site repulsion, but still with hopping of unlimited range, an unperturbed form of the model is obtained and is exactly diagonalized by Fourier transformation. Both the perturbed and unperturbed forms of the model show Bose–Einstein condensation into the lowest unperturbed single-particle state, despite the strong correlations between particles in the perturbed system. The critical temperature for condensation is always lowered by the repulsive on-site interactions. The fractional occupancy of the single-particle ground state, $\overline{n_0}(T)/n$, plays the role of an order parameter, vanishing at and beyond the critical temperature. Its value at zero temperature is unity in the unperturbed case, $1-n/N$ in the perturbed case, and it vanishes at the critical temperature with a finite slope in both cases, this slope being steeper in the perturbed case.

It may be concluded that this very particular solvable model continues to exhibit Bose–Einstein condensation even in the presence of very strong interactions between the particles, though these interactions do shift the transition to lower temperature, for all fractional occupancies of the lattice. This latter effect is different from what is found in three-dimensional hard-sphere Bose gases.

Acknowledgments

The author is grateful to Eytan Domany for suggesting that this limit of the Hubbard model might be of interest in Bose systems and for helpful discussions. The support of the Joseph and Bessie Feinberg Foundation is gratefully acknowledged.

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